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On G^2 continuous interpolatory composite quadratic Bézier curves

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Abstract

In the paper the interpolation by G^2 continuous composite quadratic Bézier curves is studied. It is shown that the interpolation problem can be naturally posed correctly in such a way that a smooth curve f is approximated up to the order 4, i.e., one order more than in the corresponding function case. In addition, the tangent direction of f is approximated up to order 3, and the curvature up to order 2.

Keywords: Quadratic Bézier curve; G^2 continuity; Interpolation; Approximation order

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1. Introduction

In [6] the following approximation problem was studied: let $T_1, T_2, \ldots, T_n$ be a given noninflecting sequence of points in a plane. Find an approximating curve B such that $B(t_i) \approx T_i$, all i, for some values of the parameter t_i. Approximating functions were taken to be piecewise quadratic parametric curves in Bézier form. It was shown that the problem can be correctly posed, and the existence and the uniqueness proved. In the conclusion some examples of extending this approach to interpolation problems were also given.

In this paper we give a rigorous analysis of the interpolation problem. The proposed scheme differs basically from the schemes that can be found in [9, 3]. As a drawback here one has to solve nonlinear rather than linear problems. However, there are two major advantages: the independence of

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parametrisation and the higher approximation order. It should also be noted that despite nonlinearity the practical applications turned out to be very efficient.

A similar interpolation problem was studied already in [8], where the interpolation at breakpoints was discussed. Forcing the interpolation points and the breakpoints together resulted in somewhat unnatural restriction of data point positions, i.e., their sequence was required to be nonreflecting. Here we release the breakpoints from also being the interpolation points. In support of this approach we mention also the fact that a similar problem in the function case, i.e., the interpolation at breakpoints by a quadratic C^1 continuous spline, is unstable [3, pp. 73–80].

To be precise, we shall study two problems. Let $f: I=[\alpha,\beta]\subset\mathbb{R}\to\mathbb{R}^2$ be a given regular smooth curve, with $d(f):=(1/\|f'\|)f'$ denoting the tangent direction, and $\kappa(f)$ the curvature. Here $\|.\|$ is the usual Euclidean norm. Assume that the curvature is nonvanishing, i.e., $|\kappa(f)|\geqslant \text{const.}>0$, on I. Extensions to the case when this condition is violated will be discussed in Sections 3 and 4. In particular, the numerical procedure will always produce a continuous piecewise quadratic interpolating Bézier curve that will be G^2 continuous in sections where the underlying curve will preserve the sign of the curvature (with the sign taking one of three possible values $\{-1,0,1\}$). Let

$$\alpha\leqslant\xi_0<\xi_1<\cdots<\xi_{n+1}\leqslant\beta$$

be a strictly increasing partition of the parameter interval.

Problem A. Let

$$T_i := f(\xi_i),\quad i=0,1,\ldots,n+1,$$
$$d_i := d(f)(\xi_i),\quad i=1,\ldots,n.$$

Find a composite quadratic G^2 continuous Bézier curve that interpolates the given data.

Problem B. Let f be additionally a closed curve, i.e., $f(\alpha)=f(\beta)$, and let

$$T_i := f(\xi_i),\quad i=1,2,\ldots,n,$$
$$d_i := d(f)(\xi_i),\quad i=1,\ldots,n.$$

Find a closed composite quadratic G^2 continuous closed Bézier curve that interpolates the given data.

The main result of the paper is:

Theorem. *There exists a* (*closed*) *quadratic n-composite Bézier curve B that solves Problem* A (*Problem* B). *If the underlying curve f is smooth enough, i.e., $f\in C^4$, the approximation order is*

$$\operatorname{dist}(f,B)=\mathcal{O}(h^4),\quad h:=\max_i \Delta\xi_i.$$

Also, for a smooth f,

$$\operatorname{dist}(d(f),d(B))=\mathcal{O}(h^3),\quad \operatorname{dist}(\kappa(f),\kappa(B))=\mathcal{O}(h^2).$$

There are two natural ways to measure the distance between curves: reparametrise one with the parameter of the other, and compare the points that belong to the common parameter values.

This leads to the parametric distance between the curves as defined in [7], obtained by the best possible reparametrisation, and the corresponding parametric approximation order. The other approach measures the distance between two curves u, v as the Hausdorff distance between two sets of points as

$$\operatorname{dist}(u,v) := \max_t \min_s ||u(t) - v(s)||$$

which is often computationally hard to obtain. The theorem will be proved for a particular reparametrisation, so it holds for both mentioned distance measures too. Also, we shall borrow the circle example from [5, 7] in order to demonstrate that $\mathcal{O}(h^4)$ is in general also the highest approximation order of this interpolation scheme.

In practical applications the derivative directions d_i are not always all known. If not, one can consider them to be free parameters available for the local Bézier curve control. One might for example determine them in such a way that the data polygon and the Bézier control polygon are as close as possible. Let $\cdot$ denote the dot product operation. Let $d_i^{\perp}$ denote the (outer) normal on f at t_i. As long as the $d_i^{\perp}$ satisfy

$$d_i^{\perp} \cdot \frac{T_{i-1}T_i}{||T_{i-1}T_i||} \geqslant \text{const.} > 0, \quad i = 1,2,\ldots,n,$$

$$d_i^{\perp} \cdot \frac{T_{i+1}T_i}{||T_{i+1}T_i||} \geqslant \text{const.} > 0, \quad i = 1,2,\ldots,n, \tag{1.1}$$

the existence and the construction work out the same way as in Problem A (Problem B). However, the approximation order is in general lowered to $O(h^3)$. Note that (1.1) is always satisfied in case d_i is actually the direction of f' at T_i since f is assumed to have nonvanishing curvature.

The proof of the theorem will be given in Section 2, the construction in Section 3, and the numerical examples in Section 4.

2. The proof of the Theorem

The proof will be given in several lemmas concerning the existence, the uniqueness and the approximation order.

Lemma 2.1. *Let T_1, T_2, T_3 be given data points and d a given direction such that*

$$d^{\perp} \cdot T_1T_2 > 0, \quad d^{\perp} \cdot T_3T_2 > 0. \tag{2.1}$$

Let

$$B : [0,1] \to R^2 \ : \ t \mapsto B(t) := b_1(1-t)^2 + b_2 2t(1-t) + b_3t^2 \tag{2.2}$$

denote a quadratic Bézier curve with control points $b_1 := T_1$, b_2, $b_3 := T_3$. Consider the problem: find b_2 such that the corresponding B would satisfy the interpolation conditions

$$B(t_1) = T_2, \quad d(B)(t_2) = d \tag{2.3}$$

for some t_1, $t_2 \in (0,1)$. It has exactly one solution iff the direction d is given at the point T_2.

Proof. Put

$$b_2 := b_2(t_1) := T_2 + \frac{1}{2t_1(1-t_1)}((T_2 - T_1)(1-t_1)^2 + (T_2 - T_3)t_1^2). \tag{2.4}$$

It is easy to verify that for such b_2 one has $B(t_1) = T_2$. The second condition in (2.3) can then be rewritten as

$$d^\perp \cdot B'(t_2) = 0. \tag{2.5}$$

Let

$$\tilde{t}_1 := \frac{1-t_1}{t_1}, \quad p_i := d^\perp \cdot T_i, \quad i = 1,2,3.$$

If $\tilde{t}_1$ is determined, so is t_1, and consequently b_2 too. Eq. (2.5) now reads

$$-2p_1(1-t_2) + 2p_2(1-2t_2) + 2p_3t_2 + \left((p_2 - p_1)\tilde{t}_1 + (p_2 - p_3)\frac{1}{\tilde{t}_1}\right)(1-2t_2) = 0 \tag{2.6}$$

and rearranged as

$$\left((p_2 - p_1)\tilde{t}_1 + (p_2 - p_3)\frac{1}{\tilde{t}_1}\right) + 2p_2(1-2t_2) = p_1(1-t_2) - 2p_3t_2. \tag{2.7}$$

If $t_2 = \frac{1}{2}$, one has infinitely many solutions $\tilde{t}_1$ if $d \parallel T_1 - T_3$, otherwise none. Assume $t_2 \neq \frac{1}{2}$. Then from (2.6)

$$\frac{p_2 - p_1}{2}\tilde{t}_1 + \frac{p_2 - p_3}{2}\frac{1}{\tilde{t}_1} = -\frac{1-t_2}{1-2t_2}(p_2 - p_1) + \frac{t_2}{1-2t_2}(p_2 - p_3). \tag{2.8}$$

Since (2.1) implies $p_2 - p_1 > 0$, $p_2 - p_3 > 0$ one has

$$\frac{p_2 - p_1}{2}\tilde{t}_1 + \frac{p_2 - p_3}{2}\frac{1}{\tilde{t}_1} \geqslant \sqrt{(p_2 - p_1)(p_2 - p_3)}$$

with equality iff

$$\tilde{t}_1 = \sqrt{\frac{p_2 - p_3}{p_2 - p_1}},$$

i.e.,

$$t_1 = \frac{\sqrt{p_2 - p_1}}{\sqrt{p_2 - p_1} + \sqrt{p_2 - p_3}}.$$

Thus we can conclude that (2.8) has a solution iff

$$-\frac{1-t_2}{1-2t_2}(p_2 - p_1) + \frac{t_2}{1-2t_2}(p_2 - p_3) \geqslant \sqrt{(p_2 - p_1)(p_2 - p_3)}. \tag{2.9}$$

A quick inspection of the left-hand side reveals that the strict inequality can be satisfied for t_2 in some subinterval of $(0,1)$. For any such t_2 (2.7) has two different solutions $\tilde{t}_1$. Thus only the equality in (2.8) produces the unique value

$$t_2 = \frac{\sqrt{p_2 - p_1}}{\sqrt{p_2 - p_1} + \sqrt{p_2 - p_3}}$$

and consequently a unique $t_1 = t_2$. □

Lemma 2.1 reveals how the interpolation should be posed, and proves the existence and the uniqueness of the solution of Problem A for the simplest case $n = 2$. The proof of the lemma also shows that

$$d^{\perp} \cdot T_1 T_2 \to 0$$

implies

$$t_1 \to 0 \quad \Longrightarrow \quad b_2 \to \lambda d \text{ and } \lambda \to \infty$$

as well as

$$d^{\perp} \cdot T_2 T_3 \to 0$$

implies that

$$t_1 \to 1 \quad \Longrightarrow \quad b_2 \to \lambda d \text{ and } \lambda \to -\infty.$$

Lemma 2.2. *There exists an n-composite quadratic G^2 continuous Bézier curve that solves Problem A. Let $(b_j)_{j=1}^{2n+1}$ be its control points, and let B_i denote the ith segment of the curve, defined by control points $b_{2i-1}, b_{2i}, b_{2i+1}$. Then B_i interpolates T_i, d_i at*

$$t_i = \frac{\sqrt{d_i^{\perp} \cdot (T_i - b_{2i-1})}}{\sqrt{d_i^{\perp} \cdot (T_i - b_{2i-1})} + \sqrt{d_i^{\perp} \cdot (T_i - b_{2i+1})}}, \quad i = 1, 2, \ldots, n, \tag{2.10}$$

and additionally

$$B_0(0) = T_0 = b_1, \quad B_n(1) = T_{n+1} = b_{2n+1}.$$

The middle control points are given by

$$b_{2i} := b_{2i}(t_i) := T_i + \frac{1}{2t_i(1-t_i)}((T_i - b_{2i-1})(1-t_i)^2 + (T_i - b_{2i+1})t_i^2). \tag{2.11}$$

Let D_{2i+1} denote the intersection of lines through T_i and T_{i+1} in directions d_i, d_{i+1} respectively, and let $\triangle_{2i+1}$ denote the triangle, defined by the lines $T_i T_{i+1}, T_i D_{2i+1}, T_{i+1} D_{2i+1}$. Then b_{2i+1} lies strictly inside the triangle $\triangle_{2i+1}$, $i = 1, 2, \ldots, n-1$.

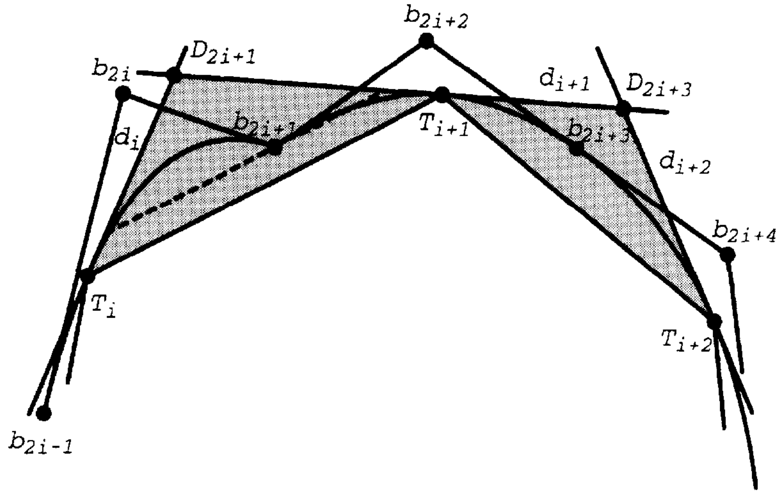

Fig. 1. The induction step of Lemma 2.2.

Proof. The proof is based on the induction (Fig. 1). Assume that for any b_{2i+1} strictly inside $\triangle_{2i+1}$ there exists a solution of Problem A that interpolates data

$$b_{2i+1}, T_{i+1}, d_{i+1}, T_{i+2}, d_{i+2}, \ldots, T_n, d_n, T_{n+1}.$$

For $i = n - 1$ this is confirmed by Lemma 2.1. Take now any fixed b_{2i-1} strictly inside $\triangle_{2i-1}$. From Lemma 2.1 one concludes that there exists a unique Bézier curve that interpolates data $b_{2i-1}, T_i, d_i, b_{2i+1}$. If one combines this curve with the solution supposed by the induction, one obtains a combined curve that interpolates the data

$$b_{2i-1}, T_i, d_i, T_{i+1}, d_{i+1}, \ldots, T_n, d_n, T_{n+1}.$$

However, at breakpoint b_{2i+1} the combined curve is only continuous. In order to complete the induction step one must show that there exists such a point b_{2i+1} at which the combined curve is also G^2 continuous. Recall the tangent direction and the curvature continuity conditions [2] at b_{2i+1}. Let ℓ_{2i+1}^- denote the line that passes through the points b_{2i}, b_{2i+1}, and ℓ_{2i+1}^+ denote the line that passes through b_{2i+1}, b_{2i+2}. The G^1 condition reads $\ell_{2i+1}^+ = \ell_{2i+1}^-$, and the G^2 condition is

$$\frac{\mathrm{dist}(b_{2i-1}, \ell_{2i+1}^+)}{||b_{2i+1} - b_{2i}||^2} = \frac{\mathrm{dist}(b_{2i+3}, \ell_{2i+1}^-)}{||b_{2i+2} - b_{2i+1}||^2}. \tag{2.12}$$

By Lemma 2.1, if b_{2i+1} approaches d_i, b_{2i} goes unboundedly along $-d_i$. Thus the left-hand side of (2.12) goes to 0, but the right-hand side stays positive. On the other hand, if b_{2i+1} approaches d_{i+1}, b_{2i+2} goes to infinity along d_{i+1}. Thus the right-hand side of (2.12) goes to 0, but the left stays positive. It follows that on any line inside $\triangle_{2i+1}$, parallel to T_iT_{i+1} there exists at least one point where (2.12) is satisfied. Now move this line continuously from T_iT_{i+1} to D_{2i+1}. From the previous discussion it follows that there exists at least one continuous curve C connecting some point on T_iT_{i+1} and D_{2i+1} on which (2.12) holds. If the point b_{2i+1} along this curve approaches D_{2i+1} one has

$$\ell_{2i+1}^- \to -d_i, \quad \ell_{2i+1}^+ \to d_{i+1}.$$

This implies

$$\angle(b_{2i+1}b_{2i+2}, b_{2i+1}b_{2i}) > \pi,$$

with the angle being measured in the positive direction. But

$$\angle(b_{2i+1}b_{2i+2}, b_{2i+1}b_{2i}) < \pi$$

near T_iT_{i+1}. Since $\ell_{2i+1}^-, \ell_{2i+1}^+$ depend continuously on b_{2i+1} there must exist on C at least one point where $\ell_{2i+1}^- = \ell_{2i+1}^+$. This brings the induction step finally to $i-1$. □

Lemma 2.3. *Let T_1, T_2, d_2, T_3 be given. Let B be the corresponding interpolatory Bézier curve. If the underlying curve f is smooth enough, i.e., $f \in C^4$, then one asymptotically has*

$$\operatorname{dist}(f, B) = \mathcal{O}(h^4),$$

where $h := \max(\Delta\xi_1, \Delta\xi_2)$. *Also*

$$\operatorname{dist}(d(f), d(B)) = \mathcal{O}(h^3), \quad f \in C^5,$$

$$\operatorname{dist}(\kappa(f), \kappa(B)) = \mathcal{O}(h^2), \quad f \in C^6.$$

Proof. Recall Lemma 2.1. Let t_1 be the parameter value at which B interpolates T_2, and b_2 given by (2.4). A direct verification yields

$$B'(t_1) = (T_2 - T_1)\frac{1-t_1}{t_1} + (T_3 - T_2)\frac{t_1}{1-t_1}. \tag{2.13}$$

Since reparametrisation of f does not change the interpolated values, one can without losing generality assume that parameter values are $\xi = (-\varepsilon, 0, \varepsilon)$, and

$$T_1 = f(-\varepsilon), \quad T_2 = f(0), \quad d_2 = d(f)(0), \quad T_3 = f(\varepsilon).$$

Here obviously $\varepsilon = \mathcal{O}(h)$. In order to estimate $f - B$ we shall reparametrise B by a parameter depending upon f. Put

$$\lambda := \frac{||f'(0)||}{||B'(t_1)||}.$$

Let t be a cubic polynomial that interpolates data $(0, t_1, \lambda, 1)$ at $(-\varepsilon, 0, 0, \varepsilon)$, i.e.,

$$t := t(\xi) := (\xi + \varepsilon)\frac{t_1}{\varepsilon} + (\xi + \varepsilon)\xi\left(\frac{\lambda}{\varepsilon} - \frac{t_1}{\varepsilon^2}\right) + (\xi + \varepsilon)\xi^2\left(\frac{1}{2\varepsilon^3} - \frac{\lambda}{\varepsilon^2}\right). \tag{2.14}$$

Note that

$$t'(\xi) = \frac{1}{2\varepsilon} + \mathcal{O}(1) \ > \ 0,$$

so t provides a regular reparametrisation as defined in [7]. Since

$$\frac{\mathrm{d}B(t(\xi))}{\mathrm{d}\xi}\Big|_{\xi=0} = \frac{\mathrm{d}B(t)}{\mathrm{d}t}\Big|_{t=t_1}\frac{\mathrm{d}t(\xi)}{\mathrm{d}\xi}\Big|_{\xi=0} = \frac{1}{\lambda}B'(t_1) = f'(0),$$

f and B now agree at the points $(-\varepsilon, 0, 0, \varepsilon)$. The fourth-order Newton expansion of $f - B$ then yields

$$f(\xi) = B(t(\xi)) + (\xi + \varepsilon)\xi^2(\xi - \varepsilon)[-\varepsilon, 0, 0, \varepsilon, \xi](f - B). \tag{2.15}$$

The last term would be $\mathcal{O}(\varepsilon^4)$ if the divided difference $[-\varepsilon, 0, 0, \varepsilon, \xi](f - B)$ could be bounded independently of ε. But f is smooth, and it is enough to consider $[-\varepsilon, 0, 0, \varepsilon, \xi]B$, i.e.,

$$\frac{\mathrm{d}^4 B(t(\xi))}{\mathrm{d}\xi^4} \tag{2.16}$$

at some interior point. Note that $B^{(3)} = 0$ and $t^{(4)} = 0$. Thus

$$\frac{\mathrm{d}^4 B(t(\xi))}{\mathrm{d}\xi^4} = B''(t)(3t''(\xi)^2 + 4t'(\xi)t^{(3)}(\xi)). \tag{2.17}$$

Under the assumption that the curvature of f does not vanish near 0 it is straightforward to obtain the following expansions:

$$\begin{aligned}
t_1 &= \tfrac{1}{2} + \mathcal{O}(\varepsilon),\\
B'(t_1) &= 2\varepsilon f'(0) + \mathcal{O}(\varepsilon^3),\\
b_2 &= f(0) + \mathcal{O}(\varepsilon^2),\\
\lambda &= \frac{1}{2\varepsilon} + \mathcal{O}(\varepsilon).
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{1}{\varepsilon^2}B''(t) &= \frac{2}{\varepsilon^2}(f(-\varepsilon) - 2b_2 + f(\varepsilon))\\
&= \frac{2}{\varepsilon^2}(f(-\varepsilon) - 2f(0) + f(\varepsilon)) + \mathcal{O}(1)\\
&= f''(\eta)\mathcal{O}(1), \quad -\varepsilon < \eta < \varepsilon,
\end{aligned}$$

and similarly for the coefficients of $\varepsilon t(\xi)$ in (2.14)

$$\begin{aligned}
&\varepsilon\left(\frac{t_1}{\varepsilon}\right) = \tfrac{1}{2} + \mathcal{O}(\varepsilon),\\
&\varepsilon\left(\frac{\lambda}{\varepsilon} - \frac{t_1}{\varepsilon^2}\right) = \varepsilon\left(\frac{1}{2\varepsilon^2} + \mathcal{O}(\varepsilon^2)\right) - \left(\frac{1}{2\varepsilon^2} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right) = \mathcal{O}(1),\\
&\varepsilon\left(\frac{1}{2\varepsilon^3} - \frac{\lambda}{\varepsilon^2}\right) = \varepsilon\left(\frac{1}{2\varepsilon^3} - \frac{1}{2\varepsilon^3} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right) = \mathcal{O}(1).
\end{aligned} \tag{2.18}$$

Thus (2.16) is bounded independently of ε. This concludes the proof of the first claim for a particular reparametrisation, so it holds for the (parametric or Hausdorff) distance between the curves too. By differentiating (2.5) one further obtains

$$\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} = B'(t)\frac{\mathrm{d}t(\xi)}{\mathrm{d}\xi} + 2\xi(2\xi^2 - \varepsilon^2)[-\varepsilon, 0, 0, \varepsilon, \xi](f - B)$$

$$+ (\xi + \varepsilon)\xi^2(\xi - \varepsilon)[-\varepsilon, 0, 0, \varepsilon, \xi, \xi](f - B). \tag{2.19}$$

From (2.17)

$$\frac{\mathrm{d}^5 B(t(\xi))}{\mathrm{d}\xi^5} = 10B''(t)t''(\xi)t^{(3)}(\xi) = \mathcal{O}(1)$$

by the previous discussion. But now $f \in C^5$, and the second divided difference in (2.19) is bounded independently of ε too. The tangent direction and the curvature do not depend on the parametrisation. So from (2.19)

$$d(B)(t) = d(B)(t(\xi)) = d(f + \mathcal{O}(\varepsilon^3))(\xi) = d(f)(\xi) + \mathcal{O}(\varepsilon^3),$$

and similarly for the curvature. □

Corollary 2.4. *The solution of Problem* A *is an* $\mathcal{O}(h^4)$ *approximation to a smooth* f.

Proof. Let f be a given smooth curve, and B its composite Bézier approximation. It is obviously possible to find a smooth curve $\tilde{f}$ which interpolates data

$$f(\xi_1), f'(\xi_1), f(\xi_2), f'(\xi_2), \ldots, f(\xi_n), f'(\xi_n)$$

and

$$b_1 = f(\xi_0), b_3, b_5, \ldots, b_{2n-1}, b_{2n+1} = f(\xi_{n+1})$$

with the curvature bounded uniformly away from 0. The latter follows from the fact that f has curvature bounded uniformly away from 0, and as a consequence the point $b_{2i+1} \in \Delta_{2i+1}$ bounded uniformly away from the boundary of Δ_{2i+1}. Thus

$$\operatorname{dist}(f, B) \leqslant \operatorname{dist}(\tilde{f}, B) + \operatorname{dist}(f, \tilde{f}) = \mathcal{O}(h^4) + \mathcal{O}(h^4) = \mathcal{O}(h^4),$$

the first by Lemma 3.2, and the second one by construction. □

Note that the order of approximation of this interpolation scheme cannot be higher in general. In particular, consider the circle $C := C(\phi) := (\cos\phi, \sin\phi)$ and data points given equidistantly,

$$\phi_i := i\phi := i\frac{2\pi}{n}, i = 1, 2, \ldots, n,$$

and

$$T_i := (\cos\phi_i, \sin\phi_i), \quad d_i := (-\sin\phi_i, \cos\phi_i), \quad i = 1, 2, \ldots, n.$$

It is simple to verify that Problem B yields independently of i,

$$t_i = \tfrac{1}{2},$$

and

$$\begin{aligned}\operatorname{dist}(B,C) &= \operatorname{dist}(B_i,C) = 1 - \tfrac{1}{2}(\tfrac{1}{2}b_{2i-1} + b_{2i} + \tfrac{1}{2}b_{2i+1})\\ &= 1 - \left(\tfrac{1}{2}\cos\frac{\phi}{2} + \tfrac{1}{2}\sqrt{1+\left(\sin\frac{\phi}{2}\right)^2}\right)\\ &= 1 - \tfrac{1}{2}\left(\sqrt{1-\left(\frac{h}{2}\right)^2} + \sqrt{1+\left(\frac{h}{2}\right)^2}\right)\\ &= \frac{1}{128}h^4 + \mathcal{O}(h^8).\end{aligned}$$

Note that Corollary 2.4 proves also the asymptotic uniqueness of the interpolatory curve.

This ends the proofs for Problem A. The other problem can be worked out by almost the same arguments.

3. The construction

Let us collect all the equations that determine the interpolatory curve together. The tangent direction continuity at the breakpoints b_{2i+1} is assured if [6]

$$b_{2i+1} = \frac{1+x_i}{2}b_{2i+2} + \frac{1-x_i}{2}b_{2i}, \quad i = 1,2,\dots,n-1, \tag{3.1}$$

with $x_i \in (-1,1)$. Further, the curvature continuity implies

$$\begin{aligned}&(1-x_{i-1})(1-x_i)^2\|b_{2i-2} - b_{2i}\|\sin\phi_i\\ &=(1+x_{i+1})(1+x_i)^2\|b_{2i+2} - b_{2i+4}\|\sin\phi_{i+1}, \quad i = 1,2,\dots,n-1,\end{aligned} \tag{3.2}$$

where $\phi_i := \angle(b_{2i}b_{2i-2}, b_{2i}b_{2i+2})$. The boundary assumptions are provided by

$$x_0 := -1, \;\; b_0 := b_1, \quad x_n := 1, \;\; b_{2n+2} := b_{2n+1}. \tag{3.3}$$

The interpolation conditions at T_i are given by (2.10) and (2.11):

$$\begin{aligned}&\tilde{t}_i = \frac{\sqrt{d_i^\perp \cdot (T_i - b_{2i+1})}}{\sqrt{d_i^\perp \cdot (T_i - b_{2i-1})}},\\ &i = 1,2,\dots,n,\\ &b_{2i} = T_i + \tfrac{1}{2}((T_i - b_{2i-1})\tilde{t}_i + (T_i - b_{2i+1})/\tilde{t}_i),\end{aligned} \tag{3.4}$$

$$b_1 = T_0, \quad b_{2n+1} = T_{n+1}. \tag{3.5}$$

Let $z := (b_{2i+1})_{i=1}^{n-1}$ be the vector of the unknown breakpoints. Given z (3.4) produces $\tilde{t}_i = \tilde{t}_i(z)$, $b_{2i} = b_{2i}(z)$. By inserting $b_{2i}(z)$ in (3.2) one obtains a nonlinear system that implicitly defines $x_i = x_i(z)$. Further, with $x_i(z), b_{2i}(z)$ one can write the right-hand side of (3.1) as

$$F(z)_i := \frac{1 + x_i(z)}{2} b_{2i+2}(z) + \frac{1 - x_i(z)}{2} b_{2i}(z), \quad i = 1, 2, \ldots, n-1.$$

By [6] and Lemma 2.2 the map F is well defined for all

$$z \in \Omega := \bigtimes_{i=1}^{n-1} \text{interior}(\Delta_{2i+1})$$

provided $\sin \phi_i(z) \sin \phi_{i+1}(z) > 0$, all i. The nonlinear system that defines the interpolatory curve can now by (3.1) be simply written as $z = F(z)$.

Several methods were applied to solve $z = F(z)$. Even the direct iteration worked quite well. However, the computationally superior and most reliable one turned out to be the numerical predictor–corrector continuation method, i.e., global homotopy method with Euler predictor and Newton corrections. A nice description of this class of methods can be found in [1]. A brief outline of the algorithm applied is as follows. Let z^0 be an initial guess of the solution suggested by Lemma 2.2,

$$z_i^0 = \frac{T_i + D_{2i+1} + T_{i+1}}{3}.$$

It is easy to verify that for h small enough $F(z^0)$ is well defined. The original function $z - F(z)$ is embedded in a global homotopy H,

$$H : \Omega \times R : (z, \lambda) \mapsto H(z, \lambda) := (z - F(z)) - \lambda(z^0 - F(z^0)).$$

The solution curve $c(s) := (z(s), \lambda(s))$, implicitly defined by $H(c(s)) = 0$, is then traced from

$$c(0) = (z(0), \lambda(0)) := (z^0, 1)$$

to $c(\tilde{s}) = (z(\tilde{s}), 0)$, the solution of the original problem. Here s denotes the natural parameter of the solution curve. Since along the curve

$$H'(c(s))\dot{c}(s) = 0,$$

the current numerical predictor step is taken in a direction t, such that $At = 0$, with A being a reasonable approximation to the Jacobian $H'(c(s))$. Corrector steps bring the estimated curve point closer to the actual curve. Corrections are of the form

$$-A^+ H,$$

where A^+ denotes the Moore–Penrose pseudo-inverse of the (approximated) Jacobian, and H is computed at the previous approximation of the current curve point. Note that asymptotically

$$\begin{aligned} F(z)_i &= \frac{b_{2i}(z) + b_{2i+2}(z)}{2} + \frac{x_i(z)}{2}(b_{2i+2}(z) - b_{2i}(z)) \\ &= \frac{b_{2i}(z) + b_{2i+2}(z)}{2} + \frac{x_i(z)}{2}\mathcal{O}(h). \end{aligned}$$

Thus

$$A \approx H'(z) = (I - F'(z); -(z^0 - F(z^0))), \tag{3.6}$$

and the approximation of F' is computed by the dependence of $x_i(z)$ on z being neglected. Note that $b_{2i}(z)$ depends only on z_{i-1}, z_i. This implies that A is a banded matrix, except for the last, constant column. This particular structure should be taken into account in QR decomposition of A, needed at each predictor step.

Each F evaluation requires solution of (3.2) denoted here simply as $G(x) = 0$. It is obvious that a global homotopy approach is a very efficient way to solve this subproblem too, since x obtained at the previous evaluation of F would be a very good initial guess.

For a closed curve interpolation only a few modifications are needed. The vector of unknowns will now be $z = (b_{2i+1})_{i=0}^{n-1}$, and a continuity condition at b_1 must be added to (3.1) and (3.2) The boundary conditions (3.3) and (3.5) are replaced by

$$x_n := x_0, \quad x_{-1} := x_{n-1},$$

$$b_{-2} := b_{2n-2}, \quad b_{-1} := b_{2n-1}, \quad b_0 := b_{2n}, \quad b_{2n+1} := b_1, \quad b_{2n+2} := b_2.$$

Some care has to be taken when the curvature condition is violated. This might be due to the fact that the underlying curve has inflection points, but also due to the roundoff errors when the curvature is close to 0. Consider first the case when f reduces to a straight line say on the segment $\xi_j, \xi_{j+1}, \ldots, \xi_i$ but not wider. This implies in (3.2)

$$\sin\phi_{j-1} \neq 0, \ \ \sin\phi_j = \sin\phi_{j+1} = \cdots = \sin\phi_i = 0, \ \ \sin\phi_{i+1} \neq 0. \tag{3.7}$$

Consequently

$$x_{j-1} = 1, \ \ x_j = x_{j+1} = \cdots = x_{i-1} = 0, \ \ x_i = -1, \tag{3.8}$$

and the remaining unknowns x_ℓ in (3.2) are determined by two separate systems of equations. If now $\sin\phi_i \sin\phi_{i+1} < 0$, for some i in (3.2), a G^2 continuous composite quadratic Bézier curve B does not exist, since B cannot change continuously in the sign of its curvature. However, let us replace the continuity of $\kappa(B)$ by the continuity of $|\kappa(B)|$, i.e., $\sin\phi_i$ in (3.2) by $|\sin\phi_i|$. This prescribes a symmetric jump of $\kappa(B)$ at the point where the curvature is discontinuous. The interpolation problem becomes correct also in the general situation, and the underlying programs reliable. Note that F is now well defined on all of Ω.

4. Examples

As the first example we consider the interpolation of the logarithmic spiral

$$f : [0, \pi] \to R^2 : \phi \mapsto f(\phi) := \ln(\phi + 1)(\cos\phi, \sin\phi).$$

Fig. 2 shows 7-composite and 11-composite Bézier interpolation, with the Bézier curve plotted dashed. The parameter partition was chosen to be equidistant. Table 1 shows the error analysis, with

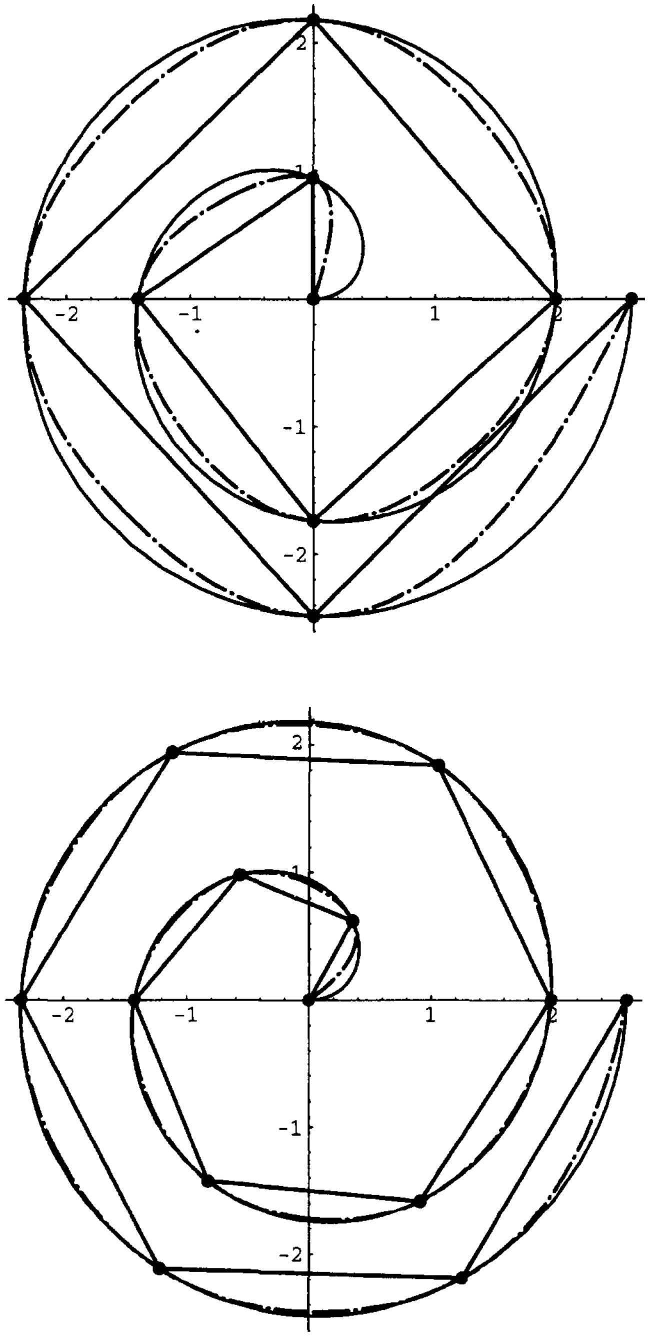

Fig. 2. The 7- and 11-composite interpolation of the spiral.

the rate gradually making its way up to 4. For simplicity, a (discrete) Hausdorff was computed, with 100 equidistant points for each curve segment.

If the directions d_i are not known one has to approximate or fix them. For example, one might require that b_{2i} lies on the symmetric bisector of the angle $\angle(T_{i-1}T_i, T_iT_{i+1})$. This produces a constant t_i,

$$t_i = \frac{\sqrt{||T_{i-1} - T_i||}}{\sqrt{||T_{i-1} - T_i||} + \sqrt{||T_i - T_{i+1}||}}.$$

Fig. 3 shows the interpolation of the spiral for such a data choice, and Table 2 the corresponding errors. The estimated rate clearly indicates the loss of one order of approximation.

As the second example we consider the interpolation of an S, i.e.,

$$f : [\tfrac{1}{4}\pi, \tfrac{7}{4}\pi] \to \mathbb{R}^2 : \phi \mapsto f(\phi) := (\sin\phi, \sin 2\phi),$$

where the curvature condition is clearly violated. The errors are given in Tables 3 and 4 (see Figs. 4 and 5).

Table 1
Errors in the interpolation of the spiral

n	Error	Rate
7	0.306466	
9	0.167296	2.71
11	0.100519	2.79
13	0.063046	3.03
15	0.041214	3.18
17	0.027839	3.33
19	0.019357	3.45

Table 2
Errors in the direction-free interpolation of the spiral

n	Error	Rate
7	0.355920	
9	0.298881	0.78
11	0.191145	2.45
13	0.129237	2.54
15	0.091222	2.61
17	0.066775	2.65
19	0.050253	2.70

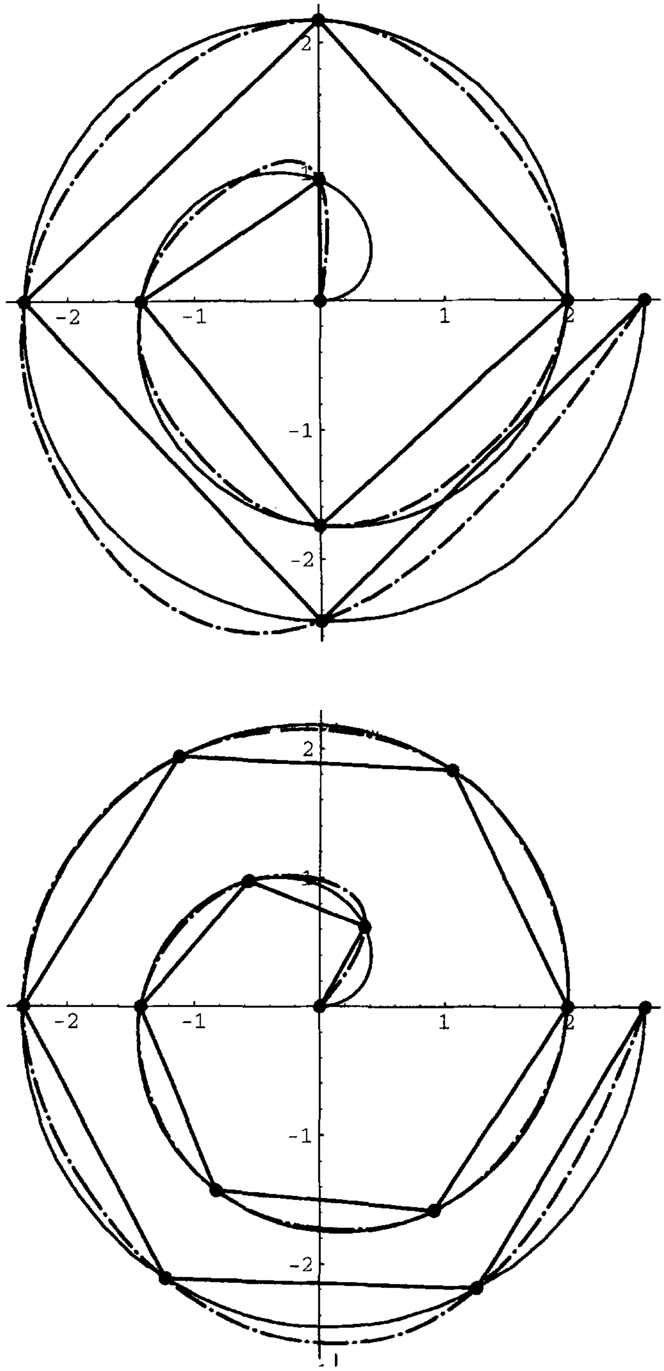

Fig. 3. The 7- and 11-composite direction-free interpolation of the spiral.

Table 3
Errors in the interpolation of the S

n	Error	Rate
7	0.0273546	
9	0.0109971	4.08
11	0.0067596	2.67
13	0.0041778	3.12
15	0.0034999	1.36
17	0.0029143	1.55
19	0.0026667	0.84

Table 4
Errors in the direction-free interpolation of the S

n	Error	Rate
7	0.0712132	
9	0.0450422	2.05
11	0.0334126	1.64
13	0.0271836	1.34
15	0.0201677	2.24
17	0.0153711	2.31
19	0.0122957	2.12

Table 5
Number of evaluations for the PC homotopy method

n	$\#H$	$\#A \approx H'$	$\#G/\#H$	$\#G'/\#H$	$\|\|\tilde{z} - F(\tilde{z})\|\|_2$
7	9	6	1.9	2.4	$4.3 \cdot 10^{-6}$
9	25	15	1.5	2.2	$3.6 \cdot 10^{-5}$
11	33	21	1.4	2.1	$8.0 \cdot 10^{-5}$
13	34	22	1.6	2.2	$1.6 \cdot 10^{-5}$
15	41	29	1.5	2.1	$1.4 \cdot 10^{-5}$
17	47	35	1.5	2.1	$9.7 \cdot 10^{-6}$
19	65	47	1.6	2.1	$9.7 \cdot 10^{-6}$

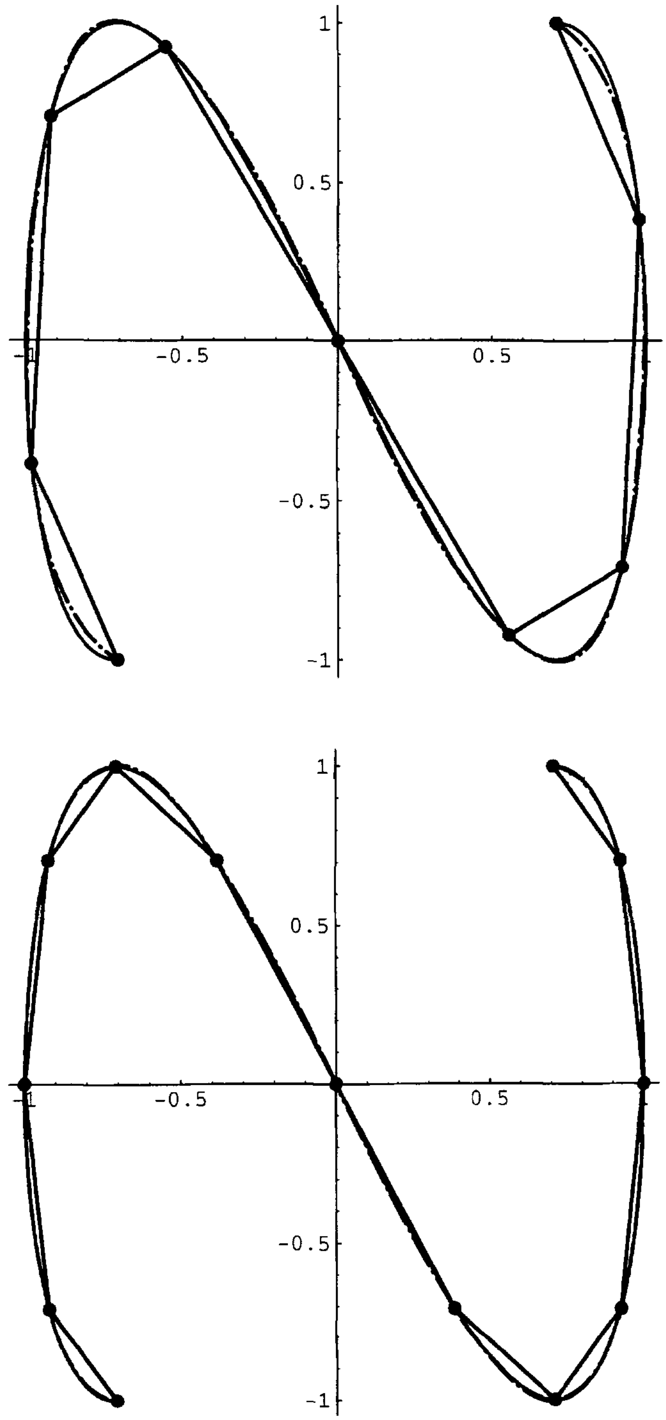

Fig. 4. The 7- and 11-composite interpolation of the S.

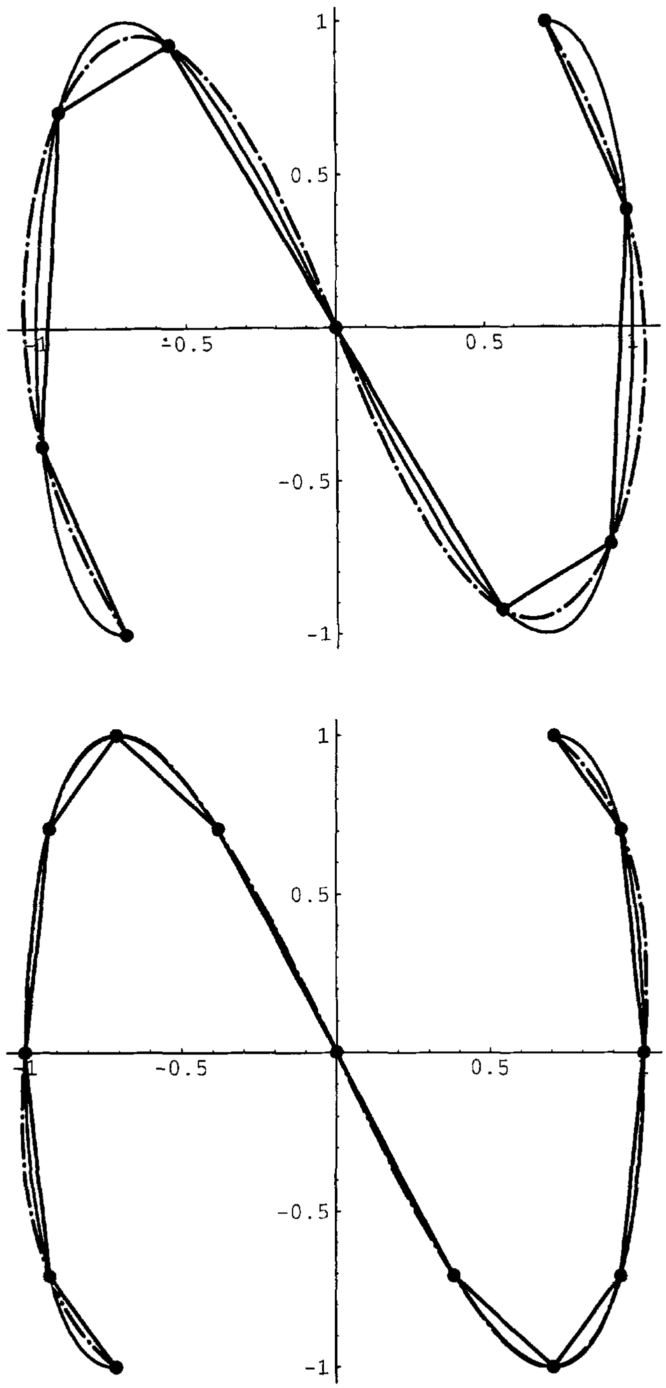

Fig. 5. The 7- and 11-composite direction-free interpolation of the S.

By comparing the corresponding errors of the tables one observes that the violation of the curvature conditions has no real impact on errors itself, but the estimated rate rather varies. Note that the interpolating curve at 0 is also G^1 continuous due to the symmetry of data. In general the curve would be only continuous with jumps in $d(B), \kappa(B)$.

At the end, we add some comments on the time complexity of numerical procedures. In order to simplify the evidence to the reader we have based the computations on the simple PC algorithm and its fortran implementation [1, pp. 266–272]. Consider the logarithmic spiral example with given directions. For the initial stepsize $\frac{1}{4}$ (1 for the subproblem $G(x)=0$), the minimal stepsize 10^{-6}, the maximal distance from the curve 0.005, the number of evaluations is given in Table 5. The second column clearly indicates that the number of evaluations of H grows only linearly with n, as well as $\#A$. Since A is banded, the QR decomposition of A takes $\mathcal{O}(n)$ too.

The next two columns show the average number of evaluations of G and G' for one H value, clearly bounded by a constant independently of n. Note that $\#G' > \#G$ is due to the evaluation of the initial tangent direction. Numerical evidence thus suggests that the overall computational complexity is $\mathcal{O}(n^2)$, with the majority of work contributed by H'.

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